A Particular Unsteady Viscometric Flow

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In a recent article Johns (*1*) treats the unsteady torsional solenoidal flow of the form

$$v^r = 0 \quad v^\theta = 0 \quad v^\phi = \omega(r, \theta, t)$$

By introducing a coordinate transformation Johns shows that this flow can be treated as a curvilinear flow, and he then solves for the components of the extra stress tensor in terms of the three viscometric functionals of the material. The coordinate transformation method could also be used for different coordinate systems.

These same results can be obtained for a general coordinate system by an alternate method. By an analysis very similar to that of Truesdell and Noll (*2*) it will be shown that any flow of the form:

$$v^1 = 0, \quad v^2 = \omega(x^1, x^3, t) \quad v^3 = 0 \tag{1}$$

with

$$h_i = h_i(x^1, x^3) \quad i = 1, 2, 3 \tag{2}$$

and

$$\frac{\partial \omega}{\partial x^1} = f(x^1, x^3)\, q(x^1, x^3, t); \quad \frac{\partial \omega}{\partial x^3} = g(x^1, x^3)\, q(x^1, x^3, t) \tag{3}$$

is a viscometric flow. The analysis will show that Equation (3) is a necessary condition for the flow to be viscometric.

The solutions for the pathlines of motion are:

$$\xi^1 = x^1, \quad \xi^2 = x^2 + \int_t^\tau \omega(x^1, x^3, \sigma)\, d\sigma, \quad \xi^3 = x^3 \tag{4}$$

The relative deformation gradient calculated from Equation (4) is

$$\|\mathbf{e}^k(\xi) \cdot \underset{\sim}{\mathbf{F}}_t(\tau)\, \mathbf{e}_m(\mathbf{x})\| = \left\| \begin{matrix} 1 & 0 & 0 \\ \dfrac{\partial}{\partial x} \displaystyle\int_t^\tau \omega(x^1, x^3, \sigma)\, d\sigma & 1 & \dfrac{\partial}{\partial x^3} \displaystyle\int_t^\tau \omega(x^1\ x^3, \sigma)\, d\sigma \\ 0 & 0 & 1 \end{matrix} \right\| \tag{5}$$

Application of Equation (3), Leibnitz's rule, and transformation to an orthonormal basis system reduces Equation (5) at $t = 0$ to

$$\|\mathbf{e}^{<k>}[\xi(\tau)] \cdot \underset{\sim}{\mathbf{F}}_{(0)}(\tau)\mathbf{e}^{<m>}(\mathbf{x})\| = \underset{\sim}{1} + \lambda(\tau) \left\| \begin{matrix} 0 & 0 & 0 \\ \nu & 0 & \mu \\ 0 & 0 & 0 \end{matrix} \right\| \tag{6}$$

where

$$\nu = \frac{\partial \omega}{\partial x^1} \frac{1}{\kappa} \frac{h_2}{h_1}, \quad \mu = \frac{\partial \omega}{\partial x^3} \frac{1}{\kappa} \frac{h_2}{h_3} \tag{7}$$

$$\kappa = h_2 \sqrt{\left(\frac{\partial \omega}{\partial x^1}\right)^2 \left(\frac{1}{h_1}\right)^2 + \left(\frac{\partial \omega}{\partial x^3}\right)^2 \left(\frac{1}{h_3}\right)^2} \tag{8}$$

and

$$\lambda(\tau) = \int_0^\tau \kappa dt \tag{9}$$

Since $\mathbf{e}^{<k>}(\xi)$ and $\mathbf{e}^{<k>}(\mathbf{x})$ are both orthonormal bases, they can be related by an orthogonal transformation,

$$\mathbf{e}^{<k>}[\xi(\tau)] = \underset{\sim}{\mathbf{Q}}(\tau)\mathbf{e}^{<k>}(\mathbf{x}) \tag{10}$$

Then Equation (6) becomes

$$\underset{\sim}{\mathbf{Q}}^T(\tau)\ \underset{\sim}{\mathbf{F}}_{(0)}(\tau) = \underset{\sim}{1} + \lambda(\tau)\underset{\sim}{\mathbf{N}} \tag{11}$$

where $\underset{\sim}{\mathbf{N}}$ with respect to the $e^{<k>}$ basis system is

$$\|\underset{\sim}{\mathbf{N}}\| = \left\| \begin{matrix} 0 & 0 & 0 \\ \nu & 0 & \mu \\ 0 & 0 & 0 \end{matrix} \right\|$$

The relative deformation gradient has the viscometric form,

$$\underset{\sim}{\mathbf{F}}_{(0)}(\tau) = \underset{\sim}{\mathbf{Q}}(\tau)\ [\underset{\sim}{1} - \lambda(\tau)\ \underset{\sim}{\mathbf{N}}_o] \tag{12}$$

where

$$\|\underset{\sim}{\mathbf{N}}_o\| = \left\| \begin{matrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \right\|$$

when $\underset{\sim}{\mathbf{N}}$ is expressed in the orthonormal basis system,

$$\begin{aligned} b^{<1>} &= \nu e^{<1>} + \mu e^{<3>} \\ b^{<2>} &= e^{<2>} \\ b^{<3>} &= -\mu e^{<1>} + \nu e^{<3>} \end{aligned} \tag{13}$$

This proves that the flow given by Equations (1), (2), and (3) is viscometric. Both ν and μ are independent of time because of Equation (3). Thus the $b^{<i>}$ coordinate system is also independent of time.

Since the flow is viscometric, the extra stress tensor can be expressed in terms of the three viscometric functionals. In the $e^{<i>}$ coordinate system the components of $\underset{\sim}{\mathbf{S}}$ are,

$$\begin{aligned} S^{<12>} &= \nu \overset{0}{\underset{s=-\infty}{\mathfrak{T}}} [\lambda(s)] \\ S^{<13>} &= \nu\mu \overset{0}{\underset{s=-\infty}{\mathfrak{S}_1}} [\lambda(s)] \\ S^{<23>} &= \mu \overset{0}{\underset{s=-\infty}{\mathfrak{T}}} [\lambda(s)] \\ S^{<11>} - S^{<33>} &= (\nu^2 - \mu^2) \overset{0}{\underset{s=-\infty}{\mathfrak{S}_1}} [\lambda(s)] \\ S^{<22>} - S^{<33>} &= \overset{0}{\underset{s=-\infty}{\mathfrak{S}_2}} [\lambda(s)] - \mu^2 \overset{0}{\underset{s=-\infty}{\mathfrak{S}_1}} [\lambda(s)] \end{aligned} \tag{14}$$

If the flow is steady Equations (14) are still valid with,

$$\begin{aligned} \lambda(s) &= \kappa S \\ \overset{0}{\underset{s=-\infty}{\mathfrak{T}}} [\lambda(s)] &= \tau(\kappa) \\ \overset{0}{\underset{s=-\infty}{\mathfrak{S}_1}} [\lambda(s)] &= \sigma_1(\kappa) \\ \overset{0}{\underset{s=-\infty}{\mathfrak{S}_2}} [\lambda(s)] &= \sigma_2(\kappa) \end{aligned} \tag{15}$$

The steady flow satisfies the criteria for a viscometric flow

given by Coleman, Markovitz, and Noll (3). Oldroyd (4) obtained essentially the same results for $S^{<ij>}$ for the steady flow case by a different method.

For the particular flow studied by Johns,

$$v^r = 0 \quad v^\phi = \omega(r, \theta, t) \quad v^\theta = 0 \tag{16}$$

$$h_1 = h_r = 1 \quad h_2 = h_\phi = r\sin\theta \quad h_3 = h_\theta = r \tag{17}$$

Then

$$\nu = \frac{r \sin\theta}{\kappa} \frac{\partial\omega}{\partial r}(r, \theta, t); \quad \mu = \frac{\sin\theta}{\kappa} \frac{\partial\omega}{\partial\theta}(r, \theta, t) \tag{18}$$

$$\kappa = r \sin\theta \sqrt{\left[\frac{\partial\omega(r, \theta, t)}{\partial r}\right]^2 + \left[\frac{\partial\omega(r, \theta, t)}{\partial\theta}\right]^2 \left(\frac{1}{r}\right)^2} \tag{19}$$

$$\lambda(s) = \int_o^s \kappa dt \tag{20}$$

Substituting Equations (18), (19), and (20) into Equation (14), we obtain Johns' equations for the extra stress tensor [his equations immediately preceding (15)]. Thus this flow pattern is curvilinear in the spherical coordinate system.

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NOTATION

$\mathbf{b}^{<i>}$ = orthonormal basis system
$\mathbf{e}^i$ = orthogonal basis system
$\mathbf{e}^{<i>}$ = orthonormal basis system
$F_{(t)}(\tau)$ = relative deformation gradient
f, g, q = separable functions of velocity gradient
g_{ij} = covariant components of the metric tensor
h_i = $\sqrt{g_{ii}}$ i = 1, 2, 3
$\mathbf{N}, \mathbf{N}_o$ = matrices defined in Equations (11) and (12)
$\tilde{\mathbf{Q}}(\tilde{\tau})$ = orthogonal matrix
r, θ, ϕ = spherical coordinates
$S^{<ij>}$ = physical components of S
$\mathbf{S}$ = extra stress tensor
$\tilde{s}$ = $t - \tau$ = time difference
t = time
v^i = contravariant components of velocity
x^i = orthogonal coordinates

Greek Letters

κ = function defined in Equation (8)
$\lambda(s)$ = function defined in Equation (9)
μ, ν = function defined in Equation (7)
ξ^i = mateiral coordinates
τ = past time
$\mathfrak{T}, \mathfrak{S}_1, \mathfrak{S}_2$ = viscometric functionals
$\tau(\kappa), \sigma_1(\kappa), \sigma_2(\kappa)$ = viscometric functions
ω = $v^2 = v^\phi$ = angular velocity

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Rectification in Continuous Contacting Equipment: On the Use of Correlations Obtained from Vaporization Experiments

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Continuous contacting equipment has frequently been employed in studies concerned with the mechanism of mass transfer in rectification. Experimental results have usually been interpreted on the assumption that the correlations for the vapor and liquid phase resistances to diffusional mass transfer in rectification are the same as, or at least similar in form to, those obtained from gas absorption or liquid vaporization experiments in the same equipment. For example, in much of the work with the popular wetted-wall column (*1* to *10*), the Chilton-Colburn (*15*) or Gilliland-Sherwood (*14*) correlations have been used as a basis in calculating the vapor film resistance.

Confining attention to rectification and liquid vaporization, a comparison of the two processes reveals several differences. Some of these, possible thermal effects (*11* to *13*) and the generally larger variation of physical properties in a rectification experiment (*7, 8, 10*), have been recognized. One effect which has not been taken into account by past experimentalists, however, is that due to the characteristic variation of the interfacial composition along the contacting equipment in rectification. In a vaporization experiment the interfacial composition may also change (due to a temperature drop resulting from evaporation of the liquid), but this change will normally be small compared to that encountered in the rectification of a mixture of high relative volatility, for example, one of the commonly used alcohol-water systems (*1* to *3, 6, 7, 8, 16*).

In view of this differing behavior at the interface, the question arises: is it permissible to use correlations obtained from vaporization work in distillation studies, even if all other differences between the two processes can be assumed to be negligible, or somehow taken into account? What kind of discrepancy may one expect between the mass transfer coefficients for rectification and vaporization as a result of this composition effect, under experimental conditions relevant to past work?

In attempting an answer, attention was focused on vapor phase mass transfer in a wetted-wall column. By using theoretical solutions of the differential equations describing the steady state mass transfer process, the Stanton group for a system in which the interfacial composition is constant was compared to this group for certain systems of varying interfacial composition. Thus, the rectification of two binary systems of constant physical properties (vapor phase Schmidt number = 0.718), with the column oper-

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